Robust strictly positive real synthesis for the fourth-order convex combinations*

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Abstract For the two fourth-order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real.

Keywords: **robust stability, strict positive realness, robust analysis and synthesis.**

The strict positive realness (SPR) of transfer functions is an important feature of a system, and plays a critical role in various fields such as absolute stability/hyperstability theory[1,2], passivity analysis[3], quadratic optimal control[4] and adaptive system theory[5]. In recent years, stimulated by the parametrization method in robust stability analysis[6,7], the robust strictly positive real systems have received much attention, and great progress has been made[8~22]. However, most results belong to the category of robust SPR analysis and valuable results in robust SPR synthesis are few. The following fundamental problem still remains open[12].

Suppose that $a(s)$ and $b(s)$ are two n-th order Hurwitz polynomials, does there exist, and how to find a (fixed) polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR?

By the definition of SPR, it is easy to know that the Hurwitz stability of the convex combination of $a(s)$ and $b(s)$ is necessary for the existence of polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. It has been proved that, if $a(s)$ and $b(s)$ have the same even (or odd) terms, such a polynomial $c(s)$ always exists[11~13], and if $n \leqslant 3$ and $a(s), b(s) \in K$(K is a stable interval polynomial set), such a polynomial $c(s)$ always exists. Recent results show that, if $n \leqslant 3$ and $a(s)$ and $b(s)$ are the two endpoints of the convex combination of stable polynomials, such a polynomial $c(s)$ always exists[16,17]. Some sufficient conditions for robust SPR synthesis have been presented in Refs. [8,15,17~19]. Especially the design method in Refs. [17] and [18] is numerically efficient for high-order polynomial segments or interval polynomials. and the conditions given

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are necessary and sufficient for low-order ($n \leq 3$) polynomial segments or interval polynomials.

It should be pointed out that, Anderson et al.[14] transformed the robust SPR synthesis problem for the fourth-order interval polynomial set into a linear programming problem in 1990 (Eqs. (58) ~ (60) in Ref. [14]), and by linear programming theory, they concluded that such a linear programming problem always has a solution. Thus, it was thought that the robust SPR synthesis problem for the fourth-order interval polynomial set had been solved. But in 1993, an example which can be synthesized showed that the corresponding linear programming problem had no solution[15]. Hence, for the fourth-order interval polynomial set, on the one hand, one cannot prove theoretically the existence of robust SPR synthesis, and on the other hand, one cannot find a counterexample that cannot be synthesized. Therefore, the robust SPR synthesis problem for an interval polynomial set, even in the case of $n = 4$, is still an open problem[11,12,14,15,17~19].

It will be shown in this paper that, for the two fourth-order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real. And the conditions given in Ref. [17] will also be shown to be necessary and sufficient for the case of the fourth-order polynomial segments.

In this paper, P^n stands for the set of n-th order polynomials of s with real coefficients, R for the field of real numbers, $\partial(p)$ for the order of polynomial $p(\cdot)$, and $H^n \subset P^n$ for the set of n-th order Hurwitz stable polynomials with real coefficients.

In the following definition, $p(\cdot) \in P^m$, $q(\cdot) \in P^n$, and $f(s) = p(s)/q(s)$ is a rational function.

Definition 1[21]. $f(s)$ is said to be strictly positive real if

(i) $\partial(p) = \partial(q)$;

(ii) $f(s)$ is analytic in $\mathrm{Re}[s] \geq 0$, (namely $q(\cdot) \in H^n$);

(iii) $\mathrm{Re}[f(j\omega)] > 0$, $\forall \omega \in \mathbb{R}$.

If $f(s) = p(s)/q(s)$ is proper, it is easy to get the following property.

Property 1[9]. If $f(s) = p(s)/q(s)$ is a proper rational function, $q(s) \in H^n$, and $\forall \omega \in R$, $\mathrm{Re}[f(j\omega)] > 0$, then $p(s) \in H^n \cup H^{n-1}$.

The following theorem is the main result of this paper.

Theorem 1. Suppose $a(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \in H^4$, $b(s) = s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4 \in H^4$, the necessary and sufficient condition for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real, is

$$\lambda b(s) + (1 - \lambda) a(s) \in H^4, \ \lambda \in [0,1].$$

Since SPR transfer functions enjoy convexity property, by Property 1, we arrive at the necessary part of the theorem.

To prove sufficiency, we must first introduce some lemmas.

Lemma 1. Suppose $a(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \in H^4$, then the following quadratic curve is an ellipse in the first quadrant of the x-y plane:

$$(a_2^2 - 4a_4)x^2 + 2(2a_3 - a_1 a_2)xy + a_1^2 y^2 - 2(a_2 a_3 - 2a_1 a_4)x - 2a_1 a_3 y + a_3^2 = 0,$$

and this ellipse is tangent with y axis at $\left(0, \frac{a_3}{a_1}\right)$, tangent with the lines $x = a_1$ and $a_3 y - a_4 x = 0$ at $\left(a_1, a_2 - \frac{a_3}{a_1}\right)$ and $\left(\frac{a_3^2}{a_2 a_3 - a_1 a_4}, \frac{a_3 a_4}{a_2 a_3 - a_1 a_4}\right)$, respectively.

Proof. Since $a(s)$ is Hurwitz stable, Lemma 1 can be proven by a direct calculation.

For notational simplicity, denote

$$\begin{aligned}\Omega_e^a := \{(x,y) \mid & (a_2^2 - 4a_4)x^2 + 2(2a_3 - a_1 a_2)xy + a_1{}^2 y^2 \\ & - 2(a_2 a_3 - 2a_1 a_4)x - 2a_1 a_3 y + a_3^2 < 0\},\\ \Omega_e^b := \{(x,y) \mid & (b_2^2 - 4b_4)x^2 + 2(2b_3 - b_1 b_2)xy + b_1^2 y^2 - 2(b_2 b_3 \\ & - 2b_1 b_4)x - 2b_1 b_3 y + b_3^2 < 0\}.\end{aligned}$$

Lemma 2. Suppose $a(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \in H^4$, $b(s) = s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4 \in H^4$, if $\lambda b(s) + (1-\lambda)a(s) \in H^4, \lambda \in [0,1]$. Then $\Omega_e^a \cap \Omega_e^b \neq \phi$.

Proof. If $\forall \lambda \in [0,1], \lambda b(s) + (1-\lambda)a(s) \in H^4$, by Lemma 1, for any $\lambda \in [0,1]$,

$$\begin{aligned}\Omega_e^{a_\lambda} := \{(x,y) \mid & (a_{\lambda 2}^2 - 4a_{\lambda 4})x^2 + 2(2a_{\lambda 3} - a_{\lambda 1} a_{\lambda 2})xy + a_{\lambda 1}^2 y^2 \\ & - 2(a_{\lambda 2} a_{\lambda 3} - 2a_{\lambda 1} a_{\lambda 4})x - 2a_{\lambda 1} a_{\lambda 3} y + a_{\lambda 3}^2 < 0\}\end{aligned}$$

is also an elliptic region in the first quadrant of the x-y plane, where $a_{\lambda i} := a_i + \lambda(b_i - a_i), i = 1, 2, 3, 4$. Apparently, when λ changes continuously from 0 to 1, $\Omega_e^{a_\lambda}$ will change continuously from Ω_e^a to Ω_e^b.

Now assume $\Omega_e^a \cap \Omega_e^b = \phi$, by Lemma 1 (without loss of generality, suppose $\frac{b_3}{b_1} > \frac{a_3}{a_1}$), $\exists\, v \in \left[\frac{a_3}{a_1}, \frac{b_3}{b_1}\right]$ and $u \neq 0$, such that the following line l

$$l: \frac{x}{u} + \frac{y}{v} = 1$$

is tangent with Ω_e^a and Ω_e^b simultaneously.

Since l is tangent with Ω_e^a, consider

$$\begin{cases} \dfrac{x}{u} + \dfrac{y}{v} = 1, \\ (a_2^2 - 4a_4)x^2 + 2(2a_3 - a_1a_2)xy + a_1^2y^2 - 2(a_2a_3 - 2a_1a_4)x - 2a_1a_3y + a_3^2 = 0, \end{cases} \tag{1}$$

and since $a(s)$ is Hurwitz stable and $u \neq 0$, by a direct calculation, we know that the necessary and sufficient condition for l being tangent with Ω_e^a is

$$uv^2 - a_1v^2 - a_2uv + a_3v + a_4u = 0. \tag{2}$$

Since l is tangent with Ω_e^b, for the same reason, we have

$$uv^2 - b_1v^2 - b_2uv + b_3v + b_4u = 0. \tag{3}$$

From (2) and (3), we obviously have, $\forall \lambda \in [0,1]$,

$$uv^2 - a_{\lambda 1}v^2 - a_{\lambda 2}uv + a_{\lambda 3}v + a_{\lambda 4}u = 0. \tag{4}$$

Eq. (4) shows that l is also tangent with $\Omega_e^{a_\lambda}$ ($\forall \lambda \in [0,1]$), but l separates Ω_e^a and Ω_e^b, and when λ changes continuously from 0 to 1, $\Omega_e^{a_\lambda}$ will change continuously from Ω_e^a to Ω_e^b, which is obviously impossible. This completes the proof.

Lemma 3. Suppose $a(s) = s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 \in H^4$, $b(s) = s^4 + b_1s^3 + b_2s^2 + b_3s + b_4 \in H^4$. If $\Omega_e^a \cap \Omega_e^b \neq \phi$, take $(x, y) \in \Omega_e^a \cap \Omega_e^b$, and let $c(s) := s^3 + xs^2 + ys + \varepsilon$ (ε is a sufficiently small positive number). Then for $\dfrac{c(s)}{a(s)}$ and $\dfrac{c(s)}{b(s)}$, we have $\forall \omega \in \mathbb{R}$, $\mathrm{Re}\left[\dfrac{c(j\omega)}{a(j\omega)}\right] > 0$ and $\mathrm{Re}\left[\dfrac{c(j\omega)}{b(j\omega)}\right] > 0$.

Proof. Suppose $(x, y) \in \Omega_e^a \cap \Omega_e^b$, and let $c(s) := s^3 + xs^2 + ys + \varepsilon$, $\varepsilon > 0$, ε sufficiently small.

For $\forall \omega \in R$, consider

$$\mathrm{Re}\left[\frac{c(j\omega)}{a(j\omega)}\right] = \frac{1}{|a(j\omega)|^2}[(a_1 - x)\omega^6 + (a_2x - a_1y - a_3)\omega^4 + (a_3y - a_4x)\omega^2 + \varepsilon(\omega^4 - a_22\omega + a_4)].$$

In order to prove that $\forall \omega \in R$, $\mathrm{Re}\left[\dfrac{c(j\omega)}{a(j\omega)}\right] > 0$, letting $t = \omega^2$, we only need to prove that, for any $\varepsilon > 0$, ε being sufficiently small, the following polynomial $f_1(t)$ satisfies

$$f_1(t) := t[(a_1 - x)t^2 + (a_2x - a_1y - a_3)t + (a_3y - a_4x)] + \varepsilon(t^2 - a_2t + a_4) > 0, \quad \forall t \in [0, +\infty).$$

Since $(x, y) \in \Omega_e^a$, by Lemma 1, $a_1 - x > 0$, $a_3y - a_4x > 0$ and $[a_2x - a_1y - a_3]^2 - 4(a_1$

$-x)(a_3y-a_4x)<0$; thus, we have

$$(a_1-x)t^2+(a_2x-a_1y-a_3)t+(a_3y-a_4x)>0,\ \forall t\in[0,+\infty).$$

Moreover, we obviously have $f_1(0)>0$. And for any $\varepsilon>0$, when t is a sufficiently large or sufficiently small positive number, we have $f_1(t)>0$. It means that there exist t_1, t_2 such that $0<t_1<t_2$, and for all $\varepsilon>0$, $t\in[0,t_1]\cup[t_2,+\infty)$, we have $f_1(t)>0$.

Denote

$$M_1=\inf_{t\in[t_1,t_2]} t[(a_1-x)t^2+(a_2x-a_1y-a_3)t+(a_3y-a_4x)],$$
$$N_1=\sup_{t\in[t_1,t_2]} |t^2-a_2t+a_4|.$$

Then $M_1>0$ and $N_1>0$. Choosing $0<\varepsilon<\frac{M_1}{N_1}$, by a direct calculation, we have

$$f_1(t)=t[(a_1-x)t^2+(a_2x-a_1y-a_3)t+(a_3y-a_4x)]+\varepsilon(t^2-a_2t+a_4)>0,$$
$$\forall t\in[0,+\infty),$$

namely

$$\forall\omega\in R,\ \mathrm{Re}\left[\frac{c(j\omega)}{a(j\omega)}\right]>0.$$

Similarly, since $(x,y)\in\Omega_e^b$, there exist t_3, t_4 such that $0<t_3<t_4$, and for all $\varepsilon>0$, $t\in[0,t_3]\cup[t_4,+\infty)$, we have $f_2(t)>0$, where

$$f_2(t):=t[(b_1-x)t^2+(b_2x-b_1y-b_3)t+(b_3y-b_4x)]+\varepsilon(t^2-b_2t+b_4).$$

Denote

$$M_2=\inf_{t\in[t_3,t_4]} t[(b_1-x)t^2+(b_2x-b_1y-b_3)t+(b_3y-b_4x)],$$
$$N_2=\sup_{t\in[t_3,t_4]} |t^2-b_2t+b_4|.$$

Then $M_2>0$ and $N_2>0$. Choosing ε such that $0<\varepsilon<\frac{M_2}{N_2}$, we have

$$\forall\omega\in R,\ \mathrm{Re}\left[\frac{c(j\omega)}{b(j\omega)}\right]>0.$$

Thus, by choosing ε such that $0<\varepsilon<\min\left\{\frac{M_1}{N_1},\frac{M_2}{N_2}\right\}$, Lemma 3 is proven.

From Theorem 2.4 in Ref. [17], or the proof of Lemma 5 in Ref. [16], we have the following lemma.

Lemma 4. Suppose $a(s)=s^4+a_1s^3+a_2s^2+a_3s+a_4\in H^4$, $b(s)=s^4+b_1s^3+b_2s^2+b_3s+b_4\in H^4$, $c(s)=s^3+xs^2+ys+z$. If $\forall\omega\in R$, $\mathrm{Re}\left[\frac{c(j\omega)}{a(j\omega)}\right]>0$ and $\mathrm{Re}\left[\frac{c(j\omega)}{b(j\omega)}\right]>0$, take

$$\tilde{c}(s) := c(s) + r \cdot d(s),\ r > 0, r \text{ sufficiently small},$$

(where $d(s)$ is an arbitrarily given monic fourth-order polynomial). Then $\frac{\tilde{c}(s)}{a(s)}$ and $\frac{\tilde{c}(s)}{b(s)}$ are both strictly positive real.

The sufficiency of Theorem 1 is proven by combining Lemmas 1 ~ 4.

Remark 1. The proof of Theorem 1 shows that we have not only proven the existence, but also provided a design method in this paper.

Remark 2. The method in this paper is also useful for high-order ($n \geqslant 5$) polynomial segments or high-order ($n \geqslant 4$) interval polynomial sets.

Remark 3. Our results can easily be generalized to discrete-time cases.

Remark 4. If $\frac{c(s)}{a(s)}$ and $\frac{c(s)}{b(s)}$ are both SPR, it is easy to show that $\forall \lambda \in [0,1]$, $\frac{c(s)}{\lambda a(s) + (1-\lambda) b(s)}$ is also SPR.

Remark 5. The stability of polynomial segments can be checked by many efficient methods, e.g. eigenvalue method, root locus method, value set method, etc.[6,7].

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